# Contributions to uncertainty caused by nonlinearity of error curves

For testing of flow meters it is assumed that the flow rate in the measurement window is constant.

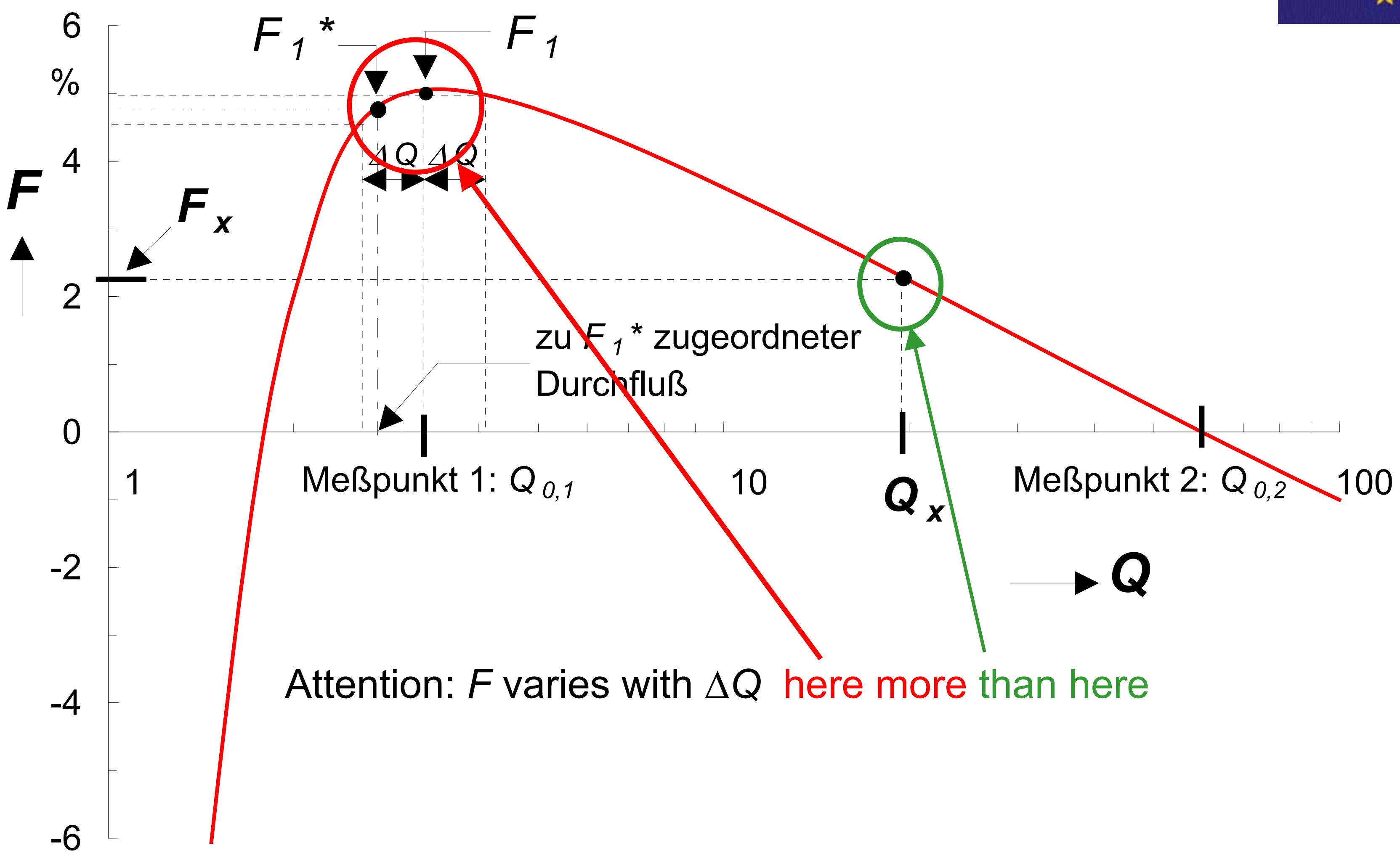
In reality the flow rate varies more or less and can be of the order 0,1 % up to 10 % (see also OIML R 49, part 2)

The question is now: how much does this effect contribute to uncertainty?

How can we solve this problem?

First: take an error curve of a water meter and see where the problematic part of the graph is!!

## Error curve of a flow meter (immoderate)



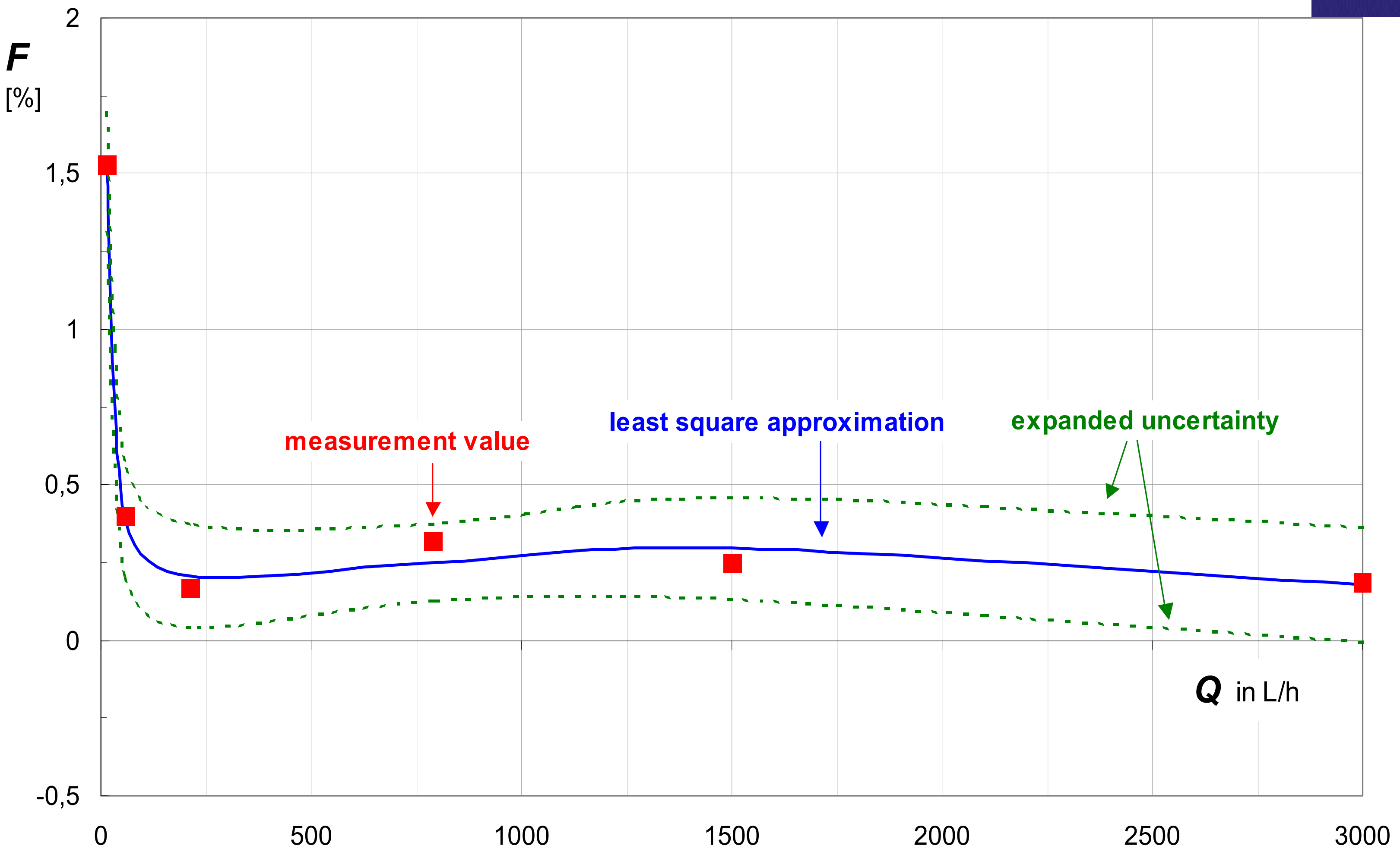
Now we have to calculate how ***F*** can vary with $\Delta Q$

## Proceeding:

1. Determine an analytical equation of the error curve
2. Afterwards calculate the contribution to uncertainty $u^2(\Delta Q)$

We will show it as an example!

# Example: characteristic error of an ultrasonic flow meter ($F(Q)$)



EMATEM

## Equation of the error curve *F*(*Q*)

From experience a good approach is given by the following equation:

$$F(Q) = a_1 + a_2\,Q + a_3\,Q^2 + \frac{a_4}{Q} + \frac{a_5}{Q^2}$$

## 2. Step: least square method (Gauss)

A short repetition of least square method by Gauss:

For the difference of the error equation ***F(Q)*** and the measurements ***F$_i$(Q$_i$)*** we get:

$$S = \sum_{i=1}^{N} \left[F_i(Q_i) - F(Q)\right]^2 \rightarrow \text{Min}$$

$$\frac{\partial S}{\partial a_i} = 0$$

$$\text{in our case: } 5 \text{ coefficients} \Rightarrow \frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = \frac{\partial S}{\partial a_3} = \frac{\partial S}{\partial a_4} = \frac{\partial S}{\partial a_5} = 0$$

# Normal equation

Condition for a minimum of the difference:

$$\frac{\partial S}{\partial a_i} = 0$$

$$\text{in our case: 5 coefficients} \Rightarrow \frac{\partial S}{\partial a_1} = \frac{\partial S}{\partial a_2} = \frac{\partial S}{\partial a_3} = \frac{\partial S}{\partial a_4} = \frac{\partial S}{\partial a_5} = 0$$

Result: Normal equation in the form: ***N a = b***

## Solution of the normal equation

$$\begin{bmatrix} N & [Q] & [Q^2] & [Q^{-1}] & [Q^{-2}] \\ [Q] & [Q^2] & [Q^3] & N & [Q^{-1}] \\ [Q^2] & [Q^3] & [Q^4] & [Q] & N \\ [Q^{-1}] & N & [Q] & [Q^{-2}] & [Q^{-3}] \\ [Q^{-2}] & [Q^{-1}] & N & [Q^{-3}] & [Q^{-4}] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} [F] \\ [F\,Q] \\ [F\,Q^2] \\ [F\,Q^{-1}] \\ [F\,Q^{-2}] \end{bmatrix}$$

or $\quad \boldsymbol{N} \quad \boldsymbol{a} = \boldsymbol{b}$

solution :

$$\boldsymbol{N}^{-1}\boldsymbol{N}\,\boldsymbol{a} = \boldsymbol{E}\,\boldsymbol{a} = \boldsymbol{a}$$

$$\boldsymbol{a} = \boldsymbol{N}^{-1}\;\boldsymbol{b}$$

remark : $[x] = \sum_i x_i$ angular brackets = sum $\Rightarrow$ Gauß´ convention

## Result for the coefficients by matrix methods

| | |
|---|---|
| $a_1$ = | 9,32E-02 |
| $a_2$ = | 2,30E-04 |
| $a_3$ = | -6,74E-08 |
| $a_4$ = | 1,40E+01 |
| $a_5$ = | 1,13E+02 |

Uncertainty of the coefficients (not the contribution of flow variations):

$u(a_1)$, $u(a_2)$, ... extract from the inverse matrix $\boldsymbol{N}^{-1}$

## Inverse matrix

Variances/ $u^2(y)$

Covariances/ $u^2(y)$

$$\boldsymbol{N}^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{bmatrix}$$

Remark: $c_{ij} = c_{ji}$ ; example: $c_{12} = c_{21}$ ...

# Coefficients of the inverse matrix - denotation

## Variances

$$u^2(a_1) = u^2(y) \times c_{11}$$
$$u^2(a_2) = u^2(y) \times c_{22}$$
$$u^2(a_3) = u^2(y) \times c_{33}$$
$$u^2(a_4) = u^2(y) \times c_{44}$$
$$u^2(a_5) = u^2(y) \times c_{55}$$

## Covariances

$$u^2(a_{12}) = u^2(y) \times c_{12} \qquad u^2(a_{13}) = u^2(y) \times c_{13}$$
$$u^2(a_{14}) = u^2(y) \times c_{14} \qquad u^2(a_{15}) = u^2(y) \times c_{15}$$
$$u^2(a_{23}) = u^2(y) \times c_{23} \qquad u^2(a_{24}) = u^2(y) \times c_{24}$$
$$u^2(a_{25}) = u^2(y) \times c_{25} \qquad u^2(a_{34}) = u^2(y) \times c_{34}$$
$$u^2(a_{35}) = u^2(y) \times c_{35} \qquad u^2(a_{45}) = u^2(y) \times c_{45}$$

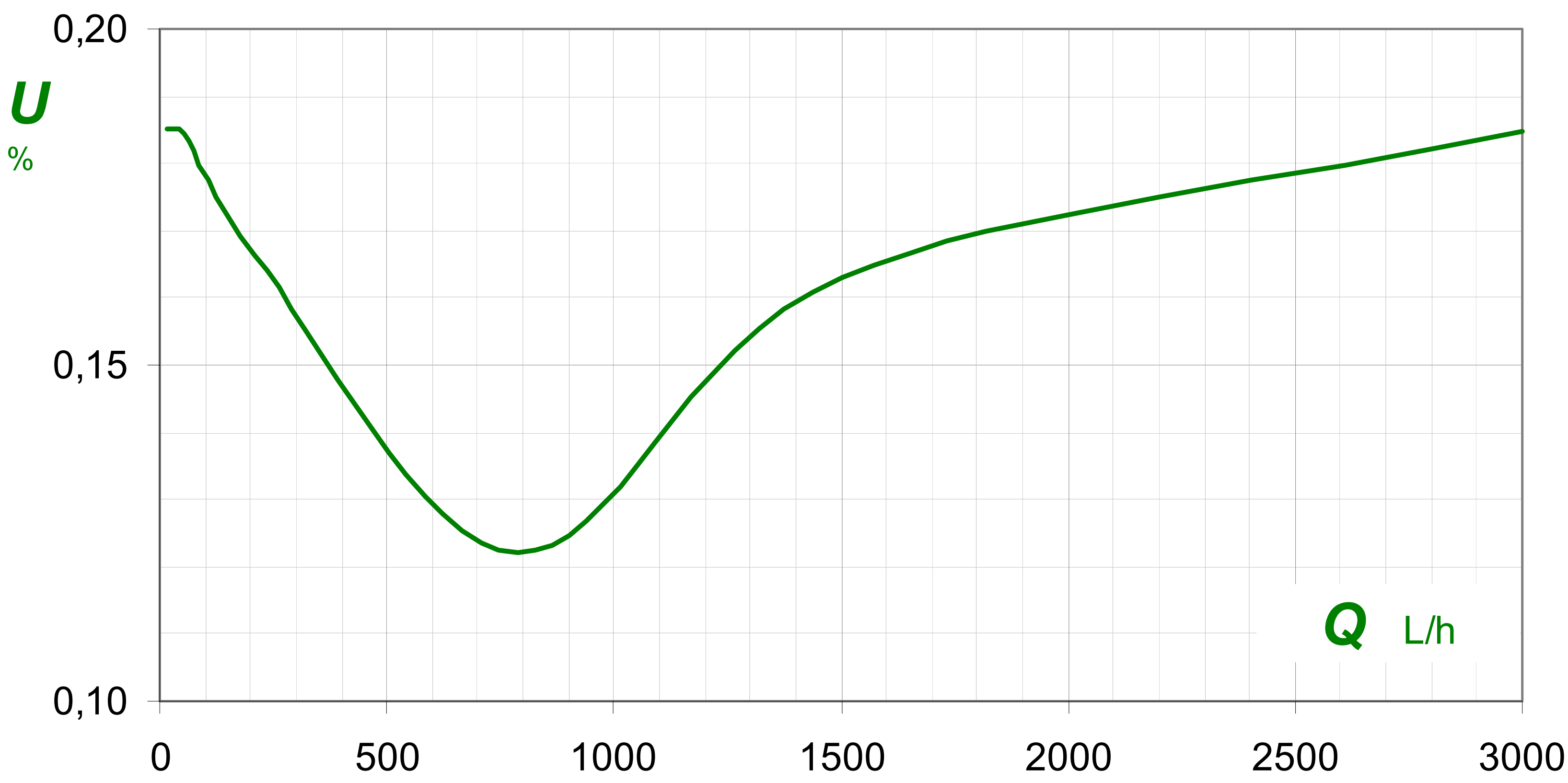
$$u^2(y) = \frac{\sum_{i=1}^{N} \left[F(Q_i) - F_i(Q_i)\right]^2}{N-5}$$

## Equation of (combined) uncertainty

$$u_c^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u(x_i, x_j)$$

$$\begin{aligned} u_c^2 = {} & u^2(a_0) + Q^2 u^2(a_1) + Q^4 u^2(a_2) + \frac{u^2(a_3)}{Q^2} + \frac{u^2(a_4)}{Q^4} + 2\,Q\,u(a_0,a_1) + \\ & + 2\,Q^2 u(a_0,a_2) + \frac{2}{Q} u(a_0,a_3) + \frac{2}{Q^2} u(a_0,a_4) + 2\,Q^3 u(a_1,a_2) + 2\,u(a_1,a_3) + \\ & + \frac{2}{Q} u(a_1,a_4) + 2\,Q\,u(a_2,a_3) + 2\,u(a_2,a_4) + \frac{2}{Q^3} u(a_3,a_4) \end{aligned}$$

## Uncertainty of the least square method (approximation)



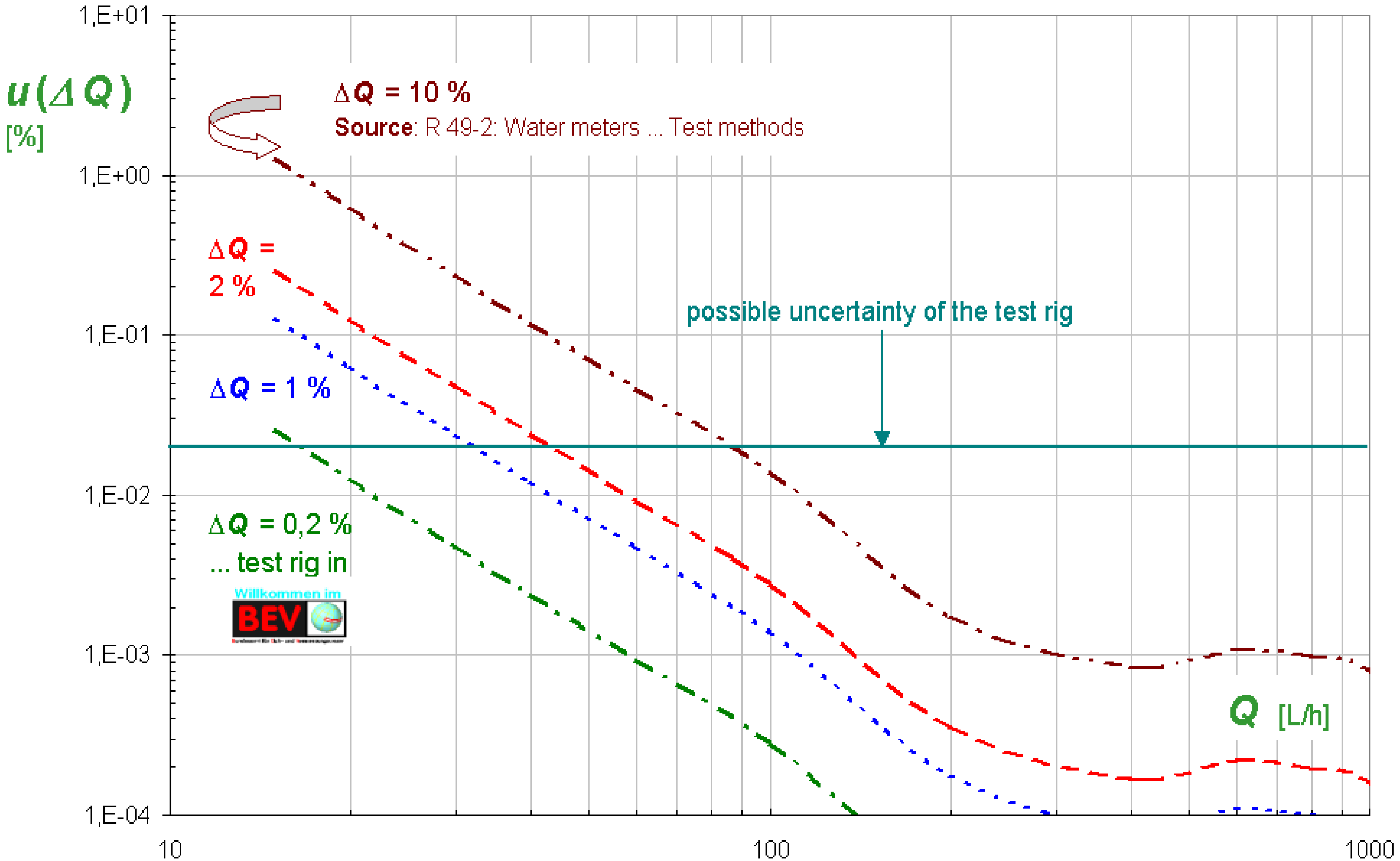
## 2nd step: the flow rate vary with $\Delta Q$

The influence of $\Delta Q$ is given by the equation: $\varepsilon = \frac{\partial F}{\partial Q}$

Next step: $u^2(Q) = \left(\frac{\partial F}{\partial Q}\right)^2 (\Delta Q)^2$

The next picture shows the result of this influence!

# Flow variations and uncertainty

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## Annex: Determination of $N^{-1}$ with Excel

**!! click the formula !!**

$$N = \begin{vmatrix} 1,32 & 3,24 \\ 5,76 & -0,72 \end{vmatrix} \qquad N^{-1} = \begin{vmatrix} 0,037 & 0,1652 \\ 0,294 & -0,067 \end{vmatrix}$$

**Solution of the equation: $x = N^{-1} y$**

$$y = \begin{vmatrix} 38,2 \\ -76 \end{vmatrix} \qquad x = N^{-1} y = \begin{vmatrix} 0,037 & 0,1652 \\ 0,294 & -0,067 \end{vmatrix} * \begin{vmatrix} 38,2 \\ -75,6 \end{vmatrix} = \begin{vmatrix} -11,09 \\ 16,31 \end{vmatrix} = \begin{matrix} x_1 \\ x_2 \end{matrix}$$