Principles of measurement uncertainty analysis

Situation

Everybody knows what the GUM is!

The main problems were: how can we combine contributions to uncertainty with different origins in a general way.

The problem is that

- systematic effects and
- random effects

are not from the same origin.

Example 1: An electric power meter consists of two - independent - parts: a measurement of the Voltage U and a measurement of the current i:

The electric power is given by the equation $P = U \times i$

This is known as the **model equation** in measurement analysis

What happens if the voltage U is affected with a systematic error ΔU and the current with a systematic error Δi?

The actual power is now:

$$P = U \times i = (U_N + \Delta U) \times (i_N + \Delta i) = U_N \times i_N + \Delta U \times i_N + \Delta i \times U_N + \Delta U \times \Delta i =$$
$$= P_N + \Delta P$$

In a different way we can write: $\Delta P = \Delta U \times i_N + \Delta i \times U_N + \Delta U \times \Delta i$

The relative error is now the ratio of ΔP and the normative power P_N

$$\frac{\Delta P}{P_N} = \frac{\Delta U \times i_N + \Delta i \times U_N + \Delta U \times \Delta i}{P_N} = \frac{\Delta U}{U_N} + \frac{\Delta i}{i_N} + \frac{\Delta U \times \Delta i}{P_N} = F_U + F_i + \Psi$$

It is the arithmetic sum of the individual contribution of the voltage measurement part and the current measurement part. At least there is also a part Ψ from quadratic order.

Is ΔU and Δi from the order 1 %, than Ψ is from the order 0,01 % ➔ in common neglectable.

$$\frac{\Delta P}{P_N} \approx \frac{\Delta U}{U_N} + \frac{\Delta i}{i_N} = F_U + F_i$$

This is the **propagation law** for systematic errors.

Propagation law for systematic errors

In general case we write this in an other way: Is ***y* = f(*x*$_1$, *x*$_2$, ...)** the result of the combination of systematic errors x_1, x_2, ... then we get for infinitesimal changes in x ➔ dx (Taylors law) for changes in x_i

$$\Delta y \approx \frac{\partial f(x\ )}{\partial x_1}\Delta x_1 + \frac{\partial f(x\ )}{\partial x_2}\Delta x_2 + ... = \sum_i \frac{\partial f(x\ )}{\partial x_i}\Delta x_i$$

Example 2: continuation of example 1:

$$\Delta P \approx \frac{\partial P}{\partial U}\Delta U + \frac{\partial P}{\partial i}\Delta i = i_N \times \Delta U + U_N \times \Delta i$$

$$\frac{\Delta P}{P} = \frac{\Delta U}{U} + \frac{\Delta i}{i} = F_u + F_i$$

What is different between random errors and systematic errors?

In the case of **random errors**, we neighter can predict the value nor the sign.

Characteristica for these random errors are:

The mean: for **random** errors with an Gaussian distribution the arithmetic mean is the **best assessment** for the true value of a quantity:

$$x_0 = \frac{1}{N}\sum_{i=0}^{N} x_i$$

➔ Details Peter Lau

An appropriate quantity for the dispersion of the measurements is the **empiric variance** of **measurements** and of the **mean**

$$u^2(y) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - x_0)^2$$

$$u(x_0)^2 = \frac{1}{N(N-1)}\sum_{i=1}^{N}(x_i - x_0)^2$$

For random errors the propagation law cannot be the same as for systematic errors, because:

- the sign of random errors is indifferent (+ or -) ,
- the value is not known

The **propagation law of Gauss** reads as follows:

$$u(y)^2 \approx \left(\frac{\partial f(x)}{\partial x_1}\right)^2 u(x_1)^2 + \left(\frac{\partial f(x)}{\partial x_2}\right)^2 u(x_2)^2 + \ldots = \sum_i \left(\frac{\partial f(x)}{\partial x_i}\right)^2 u(x_i)^2$$

$u(x_i)$ ... uncertainty contribution of the quantity x_i

$u^2(y)$ ... = $u^2_c(y)$ ... combined variance of the result = output quantity

Evaluate the random error of electrical power measurement

You remember: the model equation is $P = U \times i$

There are repeated measurements of the quantities U and i. Therefore we get means and variances for these quantities: U_0, i_0 and $u^2(U)$ and $u^2(i)$

U_0, i_0 are estimates for the quantities U and i.

The uncertainty of the output quantity is determined by the propagation law:

$$u(P)^2 = \sum_i \left(\frac{\partial P(x)}{\partial x_i}\right)^2 u(x_i)^2 = \left(\frac{\partial P}{\partial U}\right)^2 u^2(U) + \left(\frac{\partial P}{\partial i}\right)^2 u^2(i) = i^2 \times u^2(U) + U^2 \times u^2(i)$$

$$\left(\frac{u(P)}{P}\right)^2 = \frac{u^2(U)}{U^2} + \frac{u^2(i)}{i^2} = u_{r,U}^2 + u_{r,i}^2$$

Continuation of examples 1,2

Result:

The uncertainty of power measurement is: $\frac{u(P)}{P} = \sqrt{\frac{u^2(U)}{U^2} + \frac{u^2(i)}{i^2}}$

If $u(U)/U = u(i)/i = 1\ \%$ ➔ $$\frac{u(P)}{P} = \sqrt{1+1} = \sqrt{2}\ \%$$

Remember: in case of systematic errors we get for the power measurement:

$$\frac{\Delta P}{P} = 1 + 1 = 2\ \%$$

Real case

In reality the quantities are **depending** from each other - the propagation law changes as follows (two quantities):

$$u(y)^2 \approx \left(\frac{\partial f(x)}{\partial x_1}\right)^2 u(x_1)^2 + \left(\frac{\partial f(x)}{\partial x_2}\right)^2 u(x_2)^2 + 2\,\frac{\partial f(x)}{\partial x_1}\,\frac{\partial f(x)}{\partial x_2}\,u(x_1,x_2)$$

$u(x_1,x_2)$ is the covariance which is a measure for the mutual dependency of x_1 and x_2. $u(x_1,x_2)$ can be calculated from the repeated measurements!

$$u(x_1,x_2) = u(x_1)\times u(x_2)\times r(x_1,x_2)$$

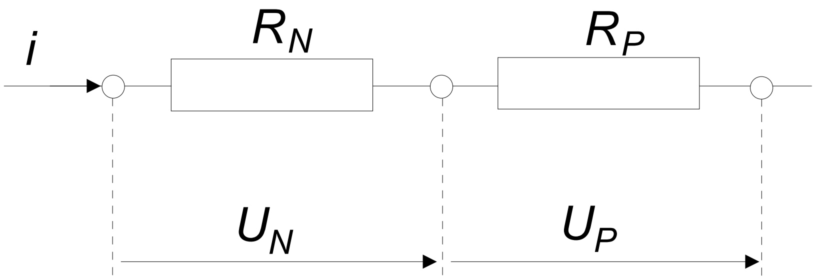
➔

$$r(x_1,x_2) = \frac{u(x_1,x_2)}{u(x_1)\times u(x_2)}$$

r is the **correlation coefficient** which varies in the range: $-1 \le r \le 1$

Example 3: Measurement of R_P by voltage measurements

Arrangement



Model equation:

$$R_P = \frac{U_P}{i} = \frac{U_P}{U_N} R_N$$

measured with the same multimeter ... correlated

from repeated measurements we get →

$U_{P,0}$ ... mean of voltage at $\boldsymbol{R}_{\mathrm{P}}$ .... 0,1025 V
$U_{N,0}$ ... mean of voltage at $\boldsymbol{R}_{\mathrm{N}}$ .... 0,1011 V

combined variance ... common propagation law

$$u_c^2(R_P) = c_{U_N}^2 u^2(U_N) + c_{U_P}^2 u^2(U_P) + c_{R_N}^2 u^2(R_N) + 2\, c_{U_N} c_{U_P} u(U_N, U_P)$$

Example 3

relative combined variance:

$$\frac{u_c^2(R_P)}{R_P^2} \approx \frac{u^2(U_N)}{U_N^2} + \frac{u^2(U_P)}{U_P^2} + \frac{u^2(R_N)}{R_N^2} - \frac{2\,u(U_N)\,u(U_P)\,r(U_N,U_P)}{U_N U_P}$$

uncertainties: $u(U_P)/U_P = 1.10^{-5}$ and $u(U_N)/U_N = 1.10^{-5}$

standard resistance: $\rightarrow$ calibration certificate

$R_N = 100\ (1 + 3{,}5.10^{-6})\ \Omega$ and $u(R_N)/R_N = 2.10^{-6}$

mean of R_P: $R_{P,0} = 101{,}385\ \Omega$

combined variance is maximal for $r = -1$: $\dfrac{u_c(R_P)}{R_P} = 2{,}01.10^{-5}$

Example 3

neglect the correlation term: $\frac{u_c(R_P)}{R_P} = 1{,}428.10^{-5}$ .... $r = 0$

result:

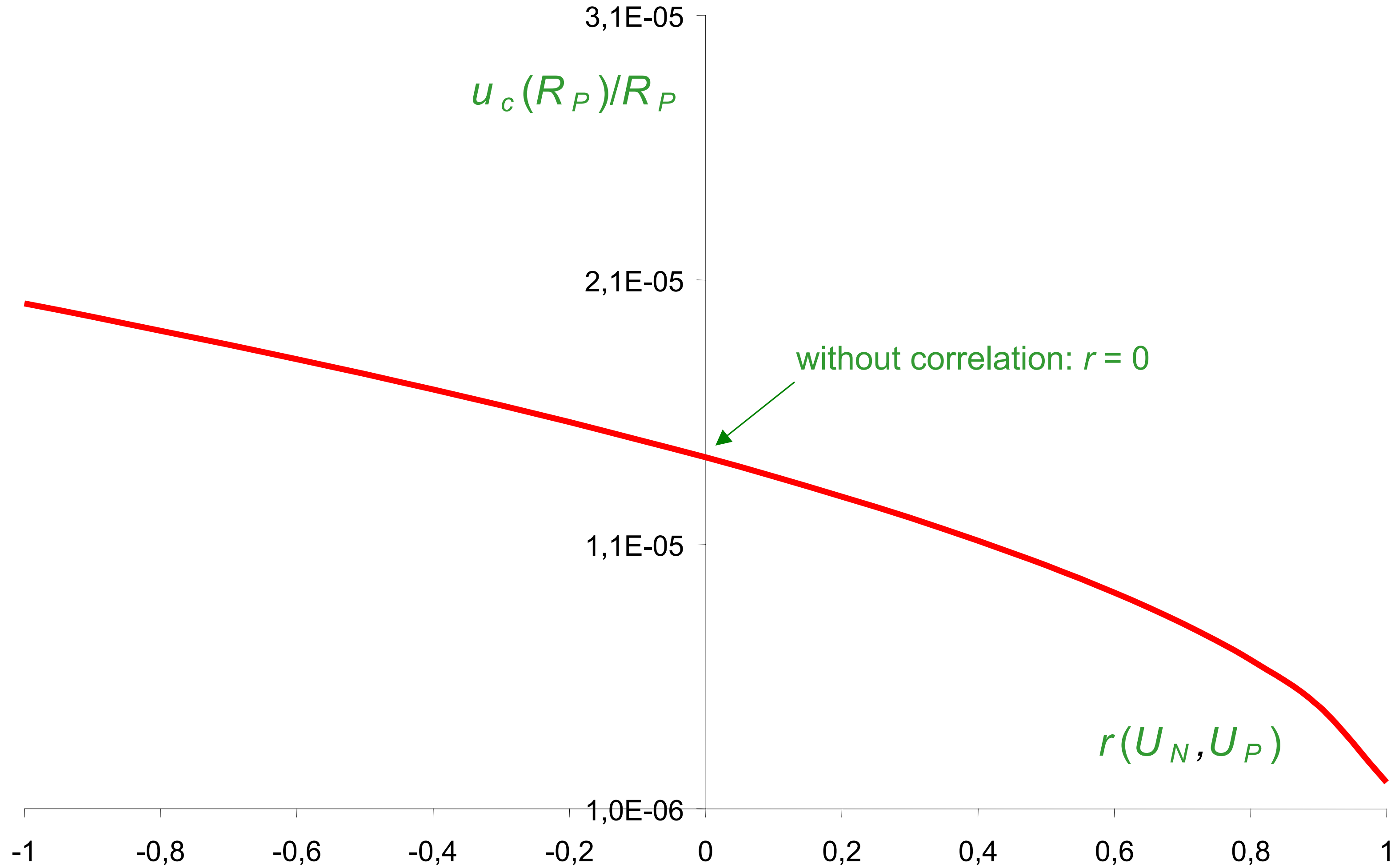
combined variance is lower by the factor $\sqrt{2}$

result for $u(R_N,R_P) = 0 \rightarrow$

$R_{P,0}$ **= 101,385 (1 ± 2.10^-5^)** oder **101,383 Ω ≤** $R_{P,0}$ **≤ 101,387 Ω**

r **= +1: 2.10^-6^:** $\frac{u_c(R_P)}{R_P} = 2.10^{-6}$

Example: influence of correlation on u_c



Analyse of the common propagation law

With the common propagation law all contributions to uncertainty are considered: in the **first part** the random errors are considered, in the **second part** also the systematic errors.

In common the uncertainty of the result is different from the uncertainty without correlation (= systematic) effects.

$$u(y)^2 \approx \left(\frac{\partial f(x)}{\partial x_1}\right)^2 u(x_1)^2 + \left(\frac{\partial f(x)}{\partial x_2}\right)^2 u(x_2)^2 + 2\,\frac{\partial f(x)}{\partial x_1}\frac{\partial f(x)}{\partial x_2}\,u(x_1, x_2)$$

Determination of the covariance, concrete example

Empirical covariance of $x_1$und x_2:

$$u(x_1,x_2)=\frac{1}{N(N-1)}\sum_{i=1}^{N}(x_{1i}-x_{1,0})(x_{2i}-x_{2,0})$$

Example 4:

Messung Nr.	x_1	x_2	$(x_{1i}-x_{1,0})$	$(x_{2i}-x_{2,0})$	$(x_{1i}-x_{1,0})*(x_{2i}-x_{2,0})$
1	1,0	0,5	-4,50	-4,33	19,49
2	2,0	0,8	-3,50	-4,03	14,11
3	3,0	-2,0	-2,50	-6,83	17,08
4	4,0	4,0	-1,50	-0,83	1,25
5	5,0	5,0	-0,50	0,17	-0,09
6	6,0	6,0	0,50	1,17	0,59
7	7,0	7,0	1,50	2,17	3,26
8	8,0	8,0	2,50	3,17	7,93
9	9,0	9,0	3,50	4,17	14,60
10	10,0	10,0	4,50	5,17	23,27
Mittelwert	5,50	4,83	0,00	0,00	101,5
Varianz	9,17	15,84			
Stababw.	3,03	3,98			
				Summe =	101,45
$u(x_1,x_2)$= Kovarianz =			$u(x_1,x_2)$= Kovarianz =	11,27	
Korrelationskoeffizient = 0,94					

There are two kinds of variances:

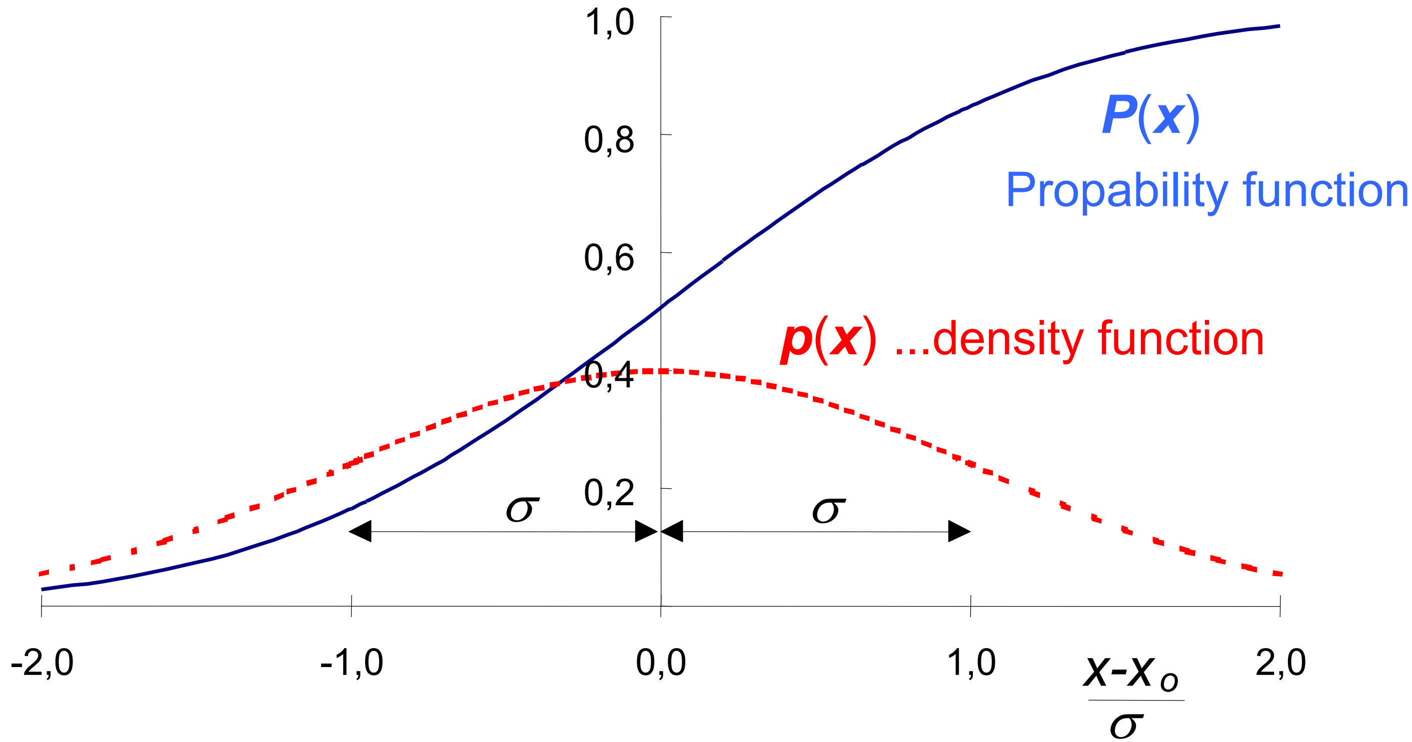
Type A: variances of repeated measurements
The formulas for this case are known

Type B: variances of other sources
For example: variances coming from former measurements, publications ...

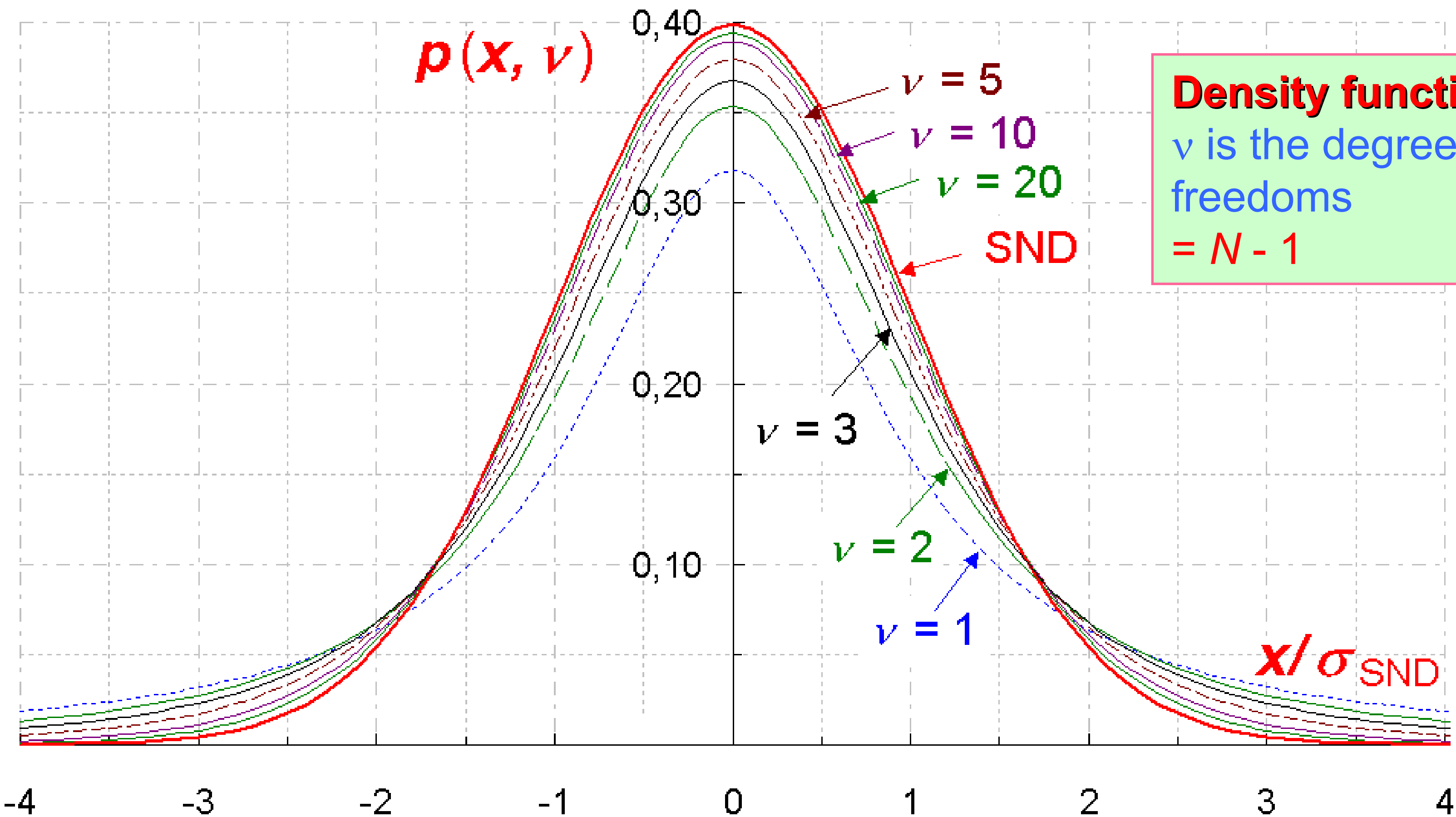
The variances of **type A and B are equivalent**. This is the most important assumption of the GUM. **Why?**

Type A variances are distributed like a **standard normal distribution** (SND, also Gauss-distribution).

The picture shows the distribution of errors for repeated measurements $N = \infty$ (infinite)



The **Student distribution** is the distribution of a sample

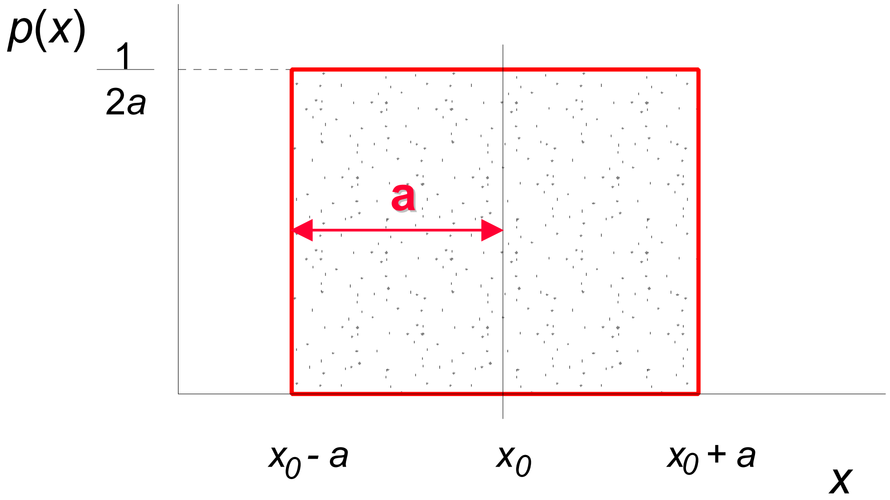


Density function
ν is the degree of freedoms
= *N* - 1

t-coefficients (Student factors)

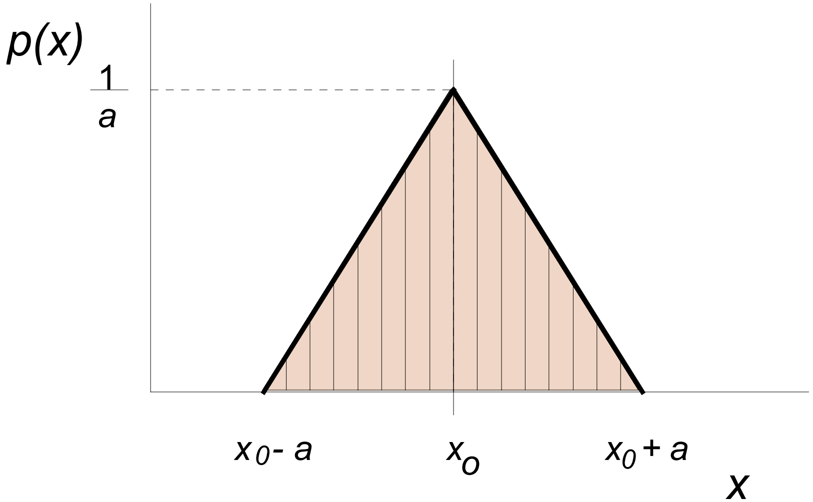
N	ν	confidential interval							$\frac{t_{95,45}}{t_{68,3}}$	$\frac{t_{99,73}}{t_{68,3}}$
		68,27	90	95	95,45	99	99.5	99,73		
2	1	1,84	6,31	12,71	13,97	63,66	127,32	235,8	7,59	128,15
3	2	1,32	2,92	4,3	4,53	9,93	14,09	19,21	3,43	14,55
4	3	1,2	2,35	3,18	3,31	5,84	7,45	9,22	2,76	7,68
5	4	1,14	2,13	2,78	2,87	4,6	5,6	6,62	2,52	5,81
6	5	1,11	2,02	2,57	2,65	4,03	4,77	5,51	2,39	4,96
8	7	1,08	1,89	2,37	2,43	3,5	4,03	4,53	2,25	4,19
10	9	1,06	1,83	2,26	2,32	3,25	3,69	4,09	2,19	3,86
20	19	1,03	1,73	2,09	2,14	2,86	3,17	3,45	2,08	3,35
50	49	1,01	1,68	2,01	2,05	2,68	2,94	3,16	2,03	3,13
100	99	1,005	1,66	1,984	2,025	2,626	2,935	3,077	2,01	3,06
1000	999	1,0000	1,65	1,965	2,005	2,585	2,89	3,015	2,01	3,02
10000	9999	1,0000	1,645	1,96	2,0000	2,576	2,866	3,0000	2,00	3,00

If there is no information about a quantity but only about the error limits we get a **rectangular distribution**:



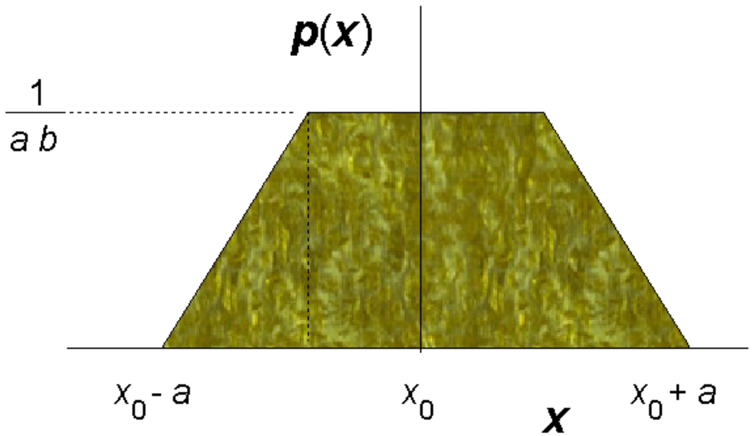
It is the most frequent distribution in uncertainty analysis

The variance of this distribution is: $u^2(x) = \frac{a^2}{3}$ or $u(x) = \frac{a}{\sqrt{3}}$



Triangle distribution

$$\sigma^2 = \frac{a^2}{6}$$



Trapezoid distribution

something between a rectangular and a triangle distribution

Remark: σ is the variance and not an estimate, why? ➔ homework

What is the distribution of the measurement result ?

Ideal situation (capital letters)

$Y(X) = f(X_1, X_2, \ldots X_n)$ ➔ relation between the **input quantities**

SND

rectangular

triangle

Real Situation

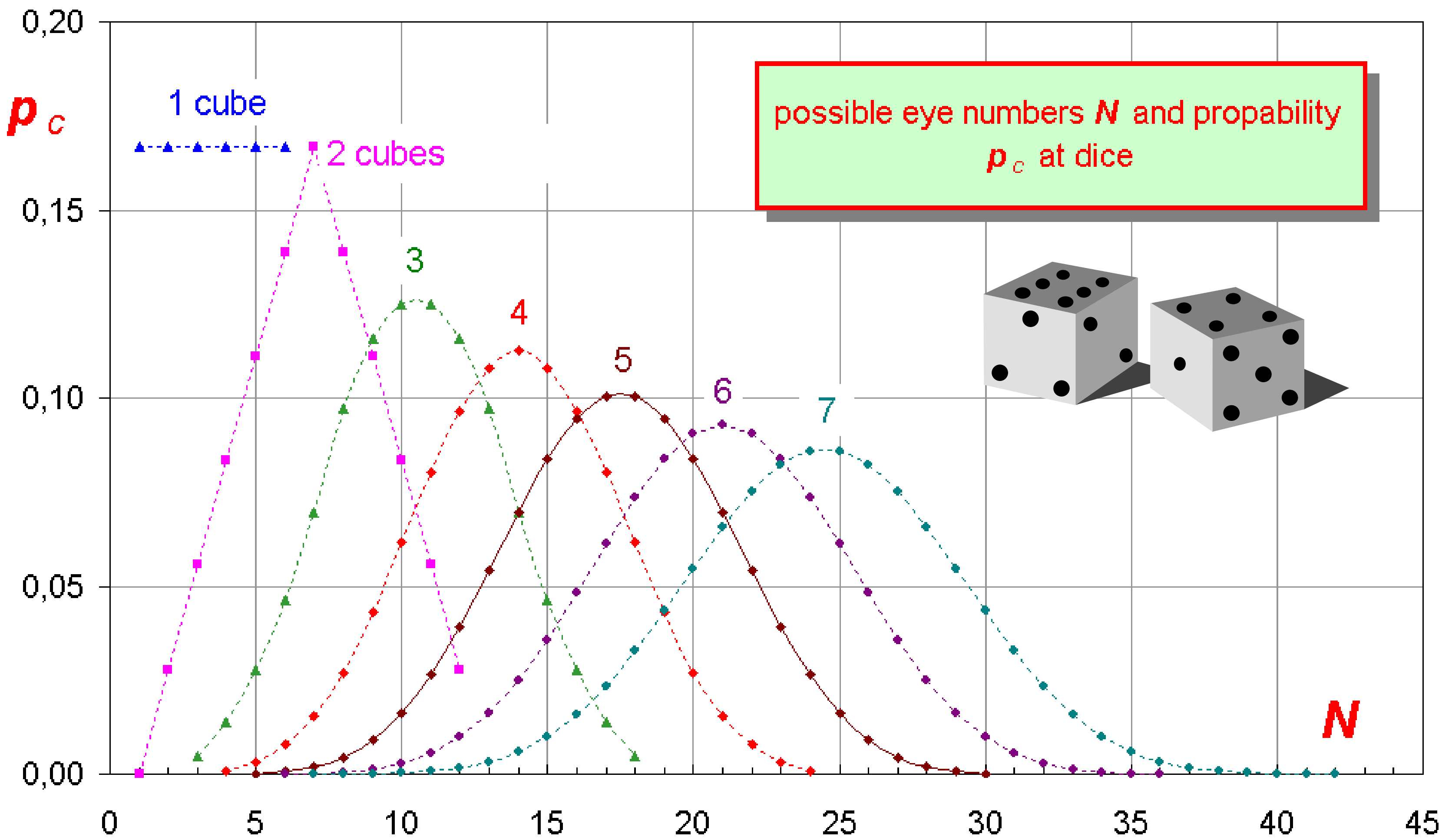
$y(x) = f(x_1, x_2, \ldots x_n)$ ➔ relation between the **estimations of the quantities**

Student (t)

trapezoid

nearly triangle

An Example: Dice with one or more cubes (discrete distr.)



Principle of convolution („Faltung“)

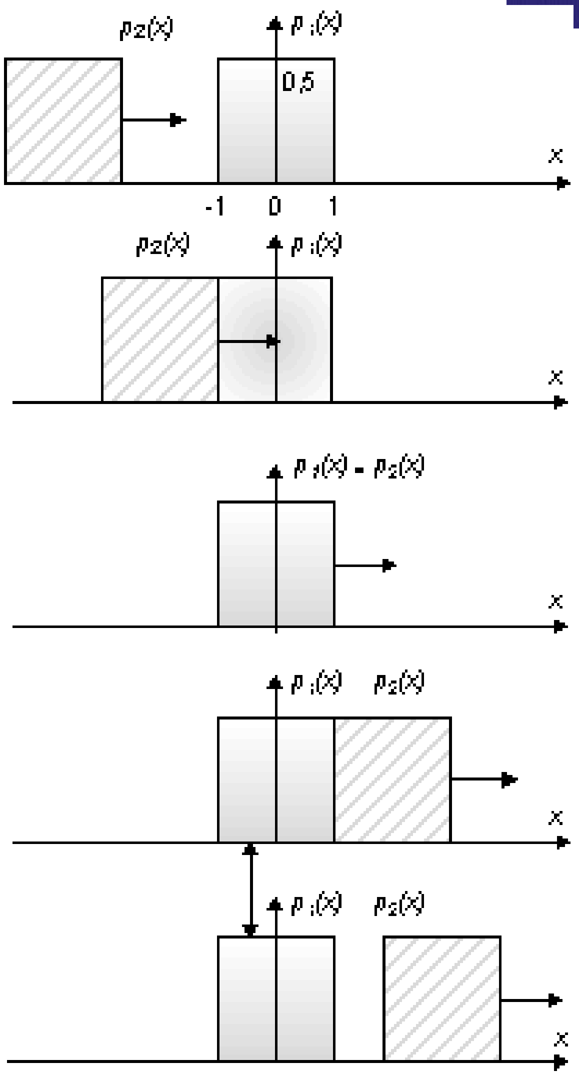
Last picture expresses a discrete distribution: only specific numbers are possible.

For **continuous distributions** we get functions instead of numbers. The principle of convolution is the same.

First step: Get a rectangular distribution and shift it over an other fixed rectangular distribution.

Second step: evaluate for all ***x*** the function:

$$p_1 \otimes p_2(x^*) = \int_{-\infty}^{\infty} p_1(x) p_2(x^* - x)\, dx$$



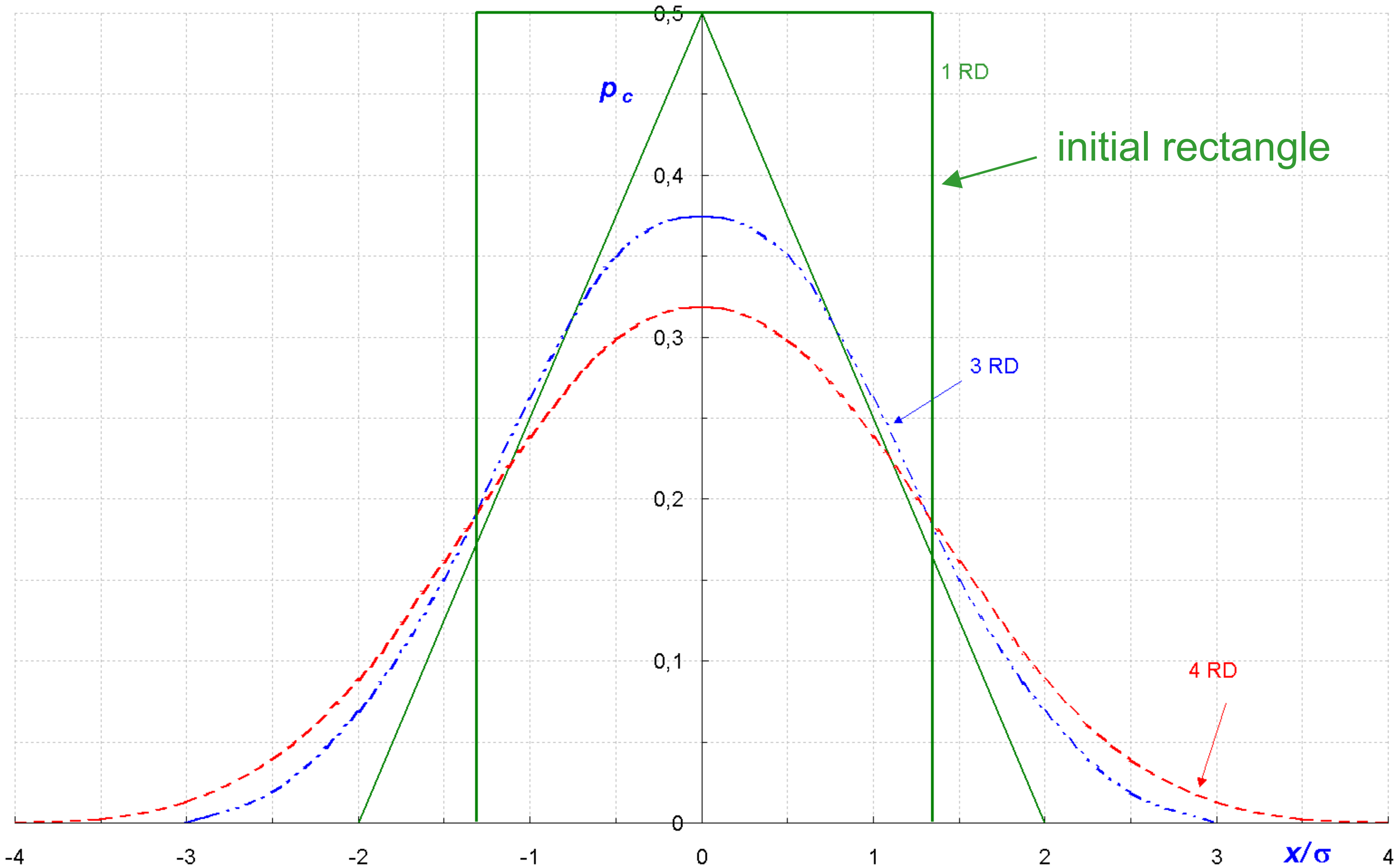
We will demonstrate the **convolution function** of

- **rectangular distributions**

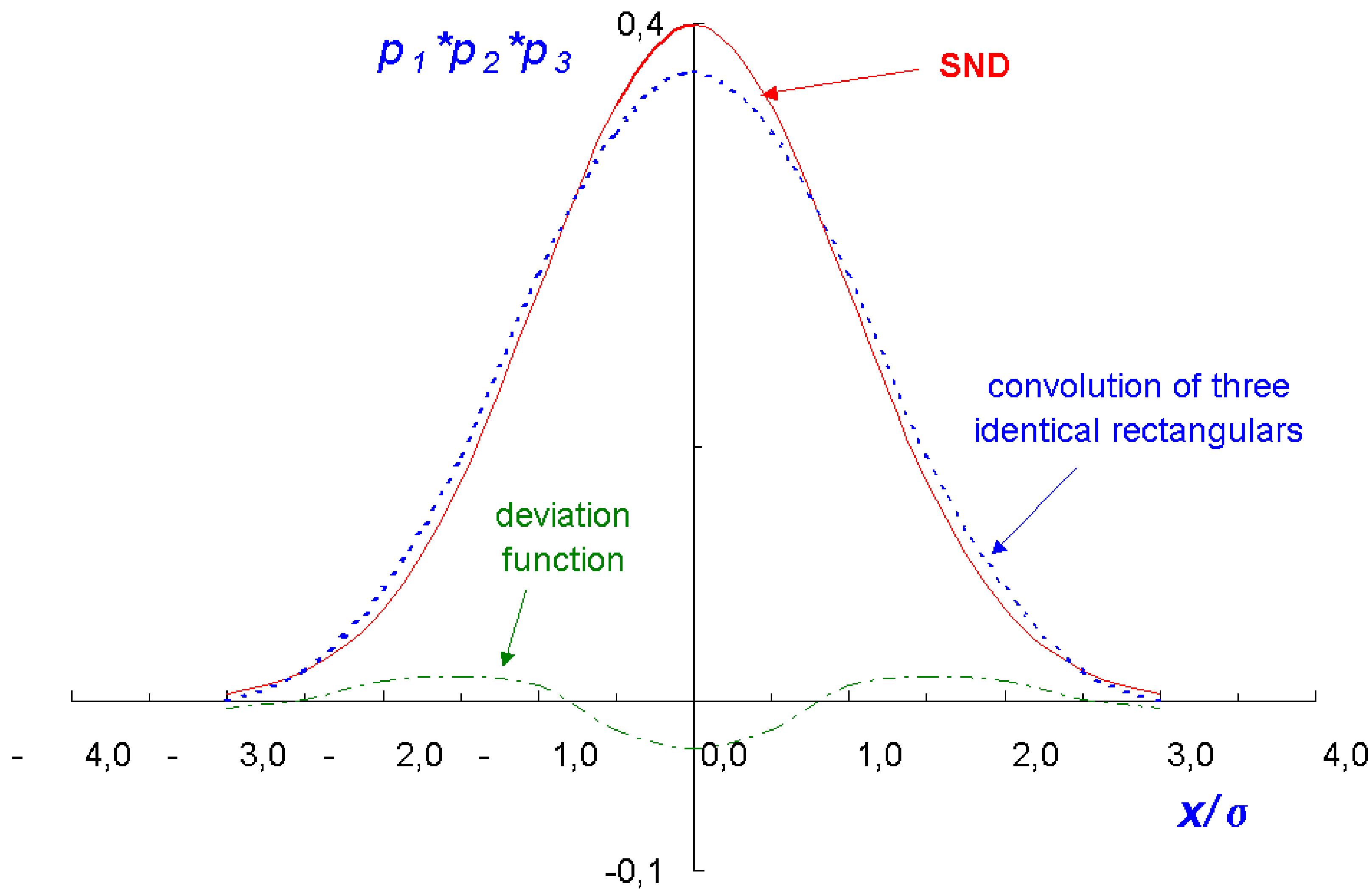
and

- **SND and rectangulars**

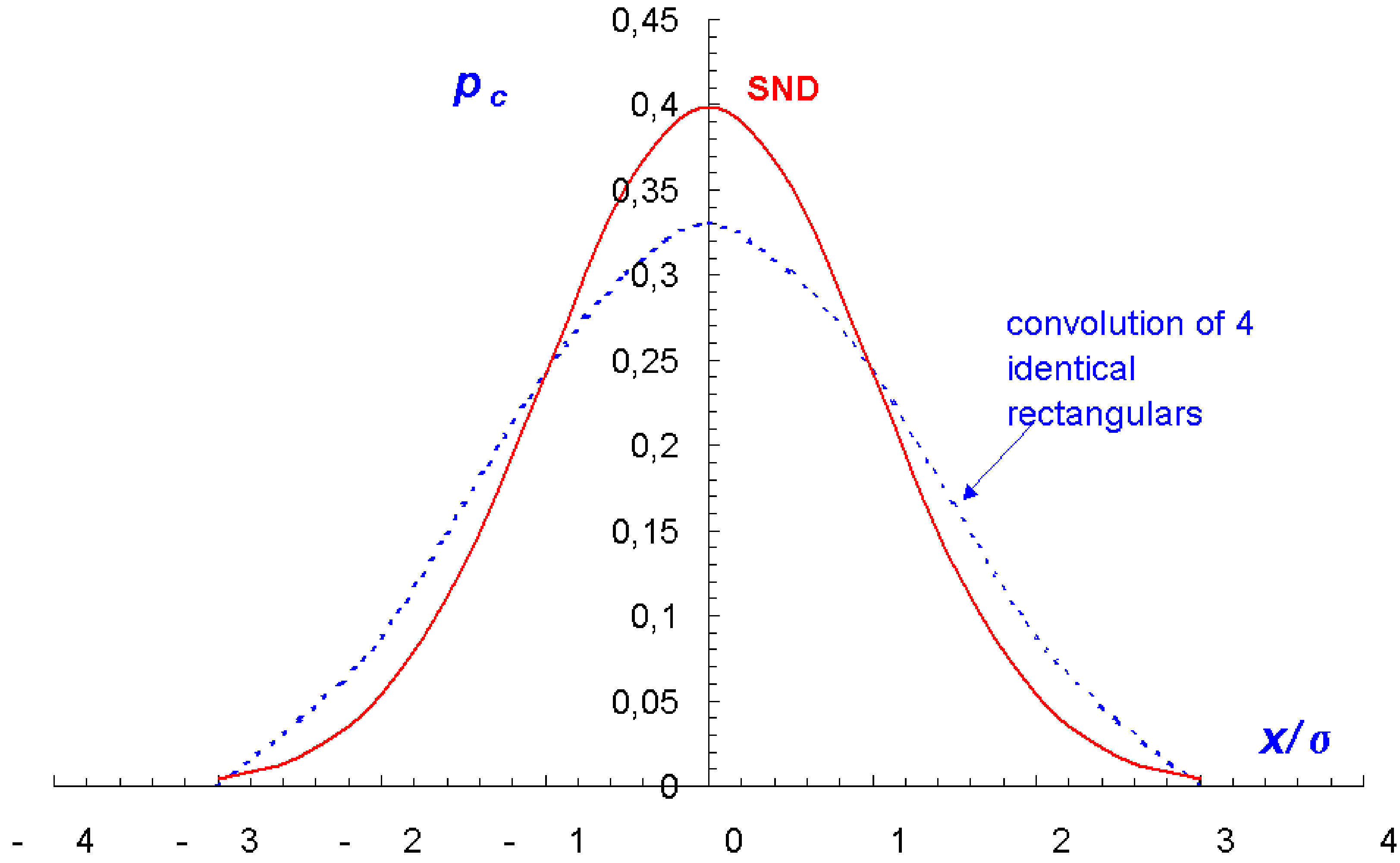
Convolution of some identic rectangular distributions



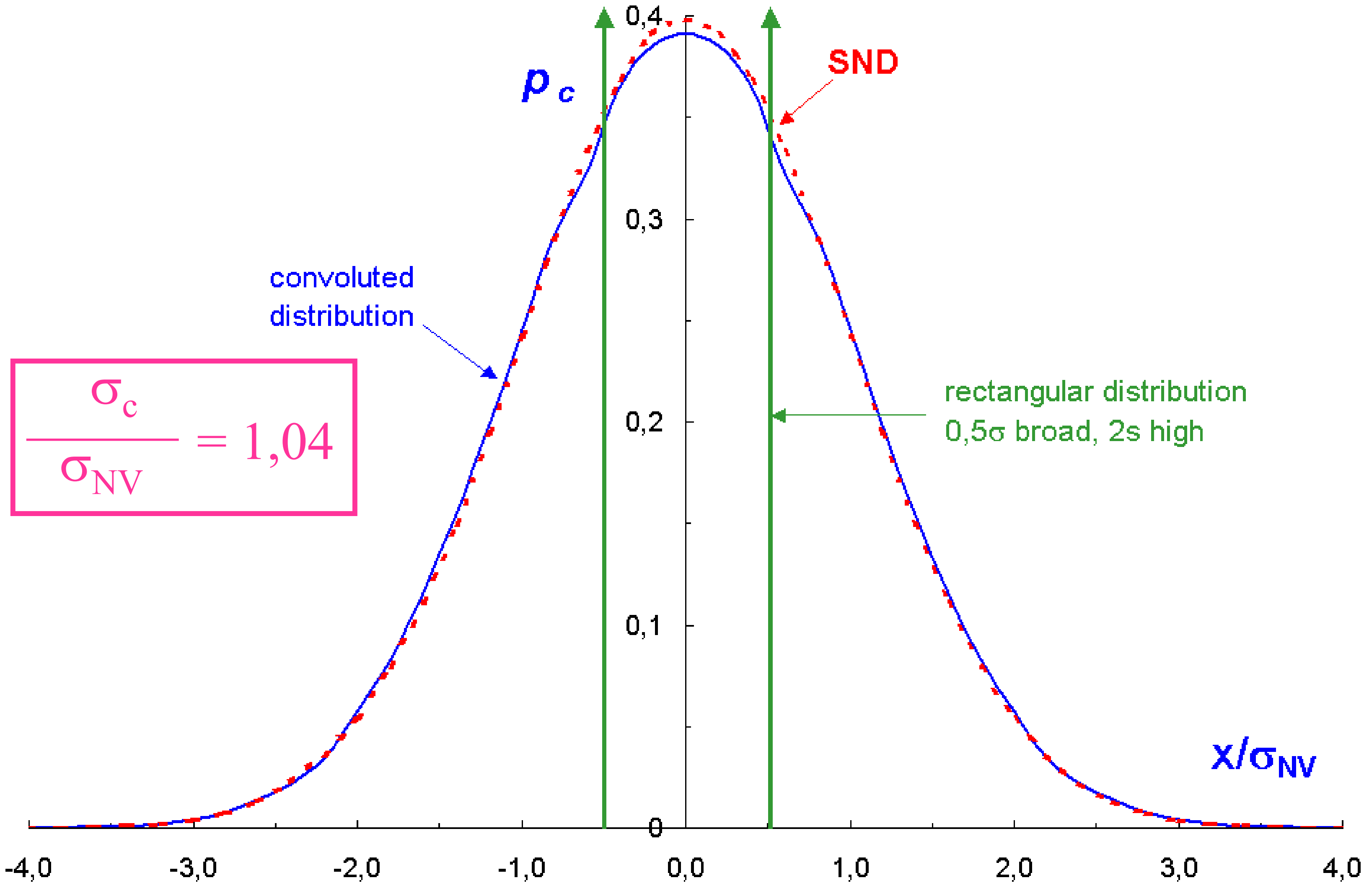
Convolution of three rectangulars in comparison with SND



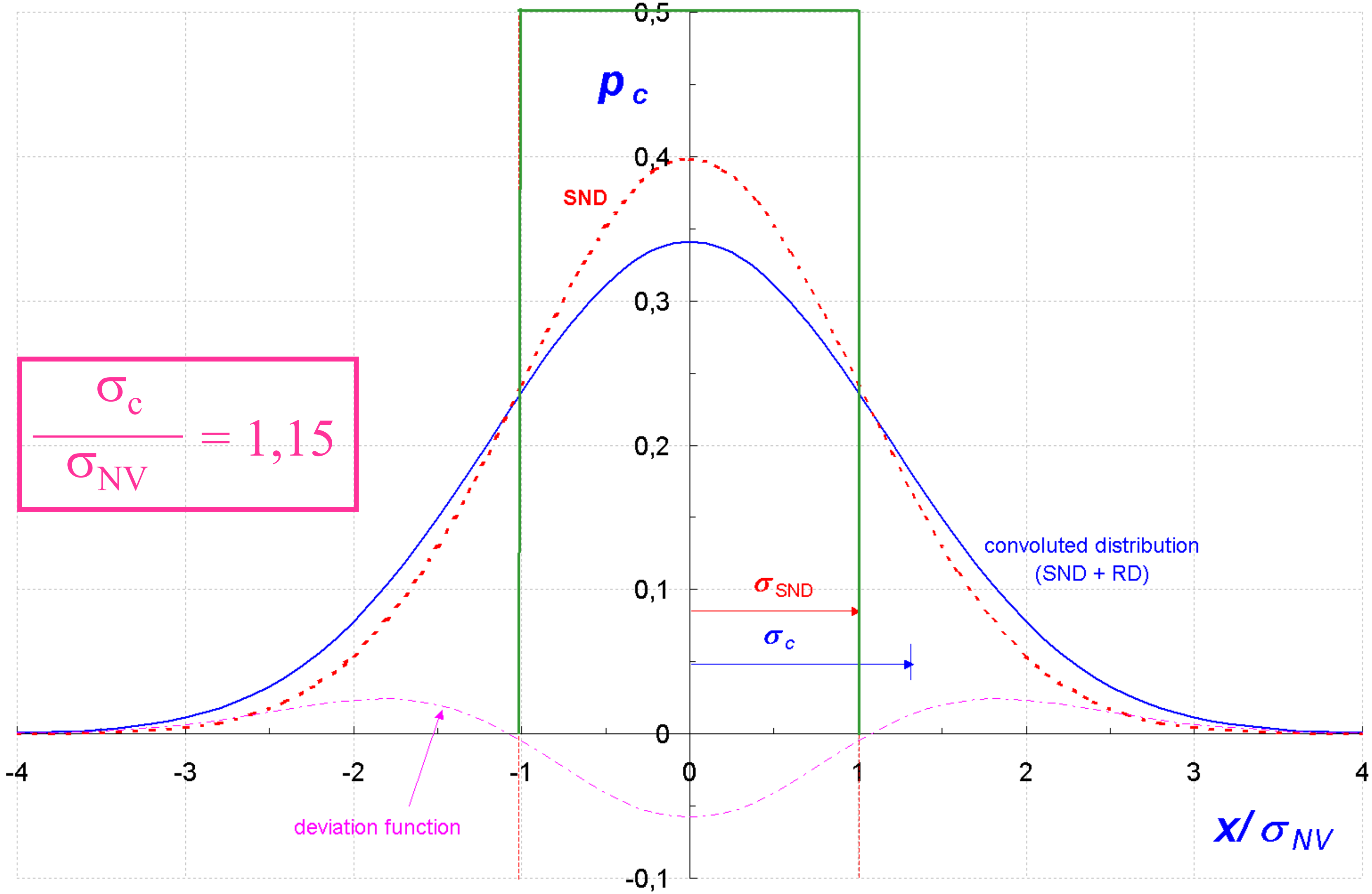
Convolusion of four identical rectangulars



Convolution of a SND and a very small rectangular distr.

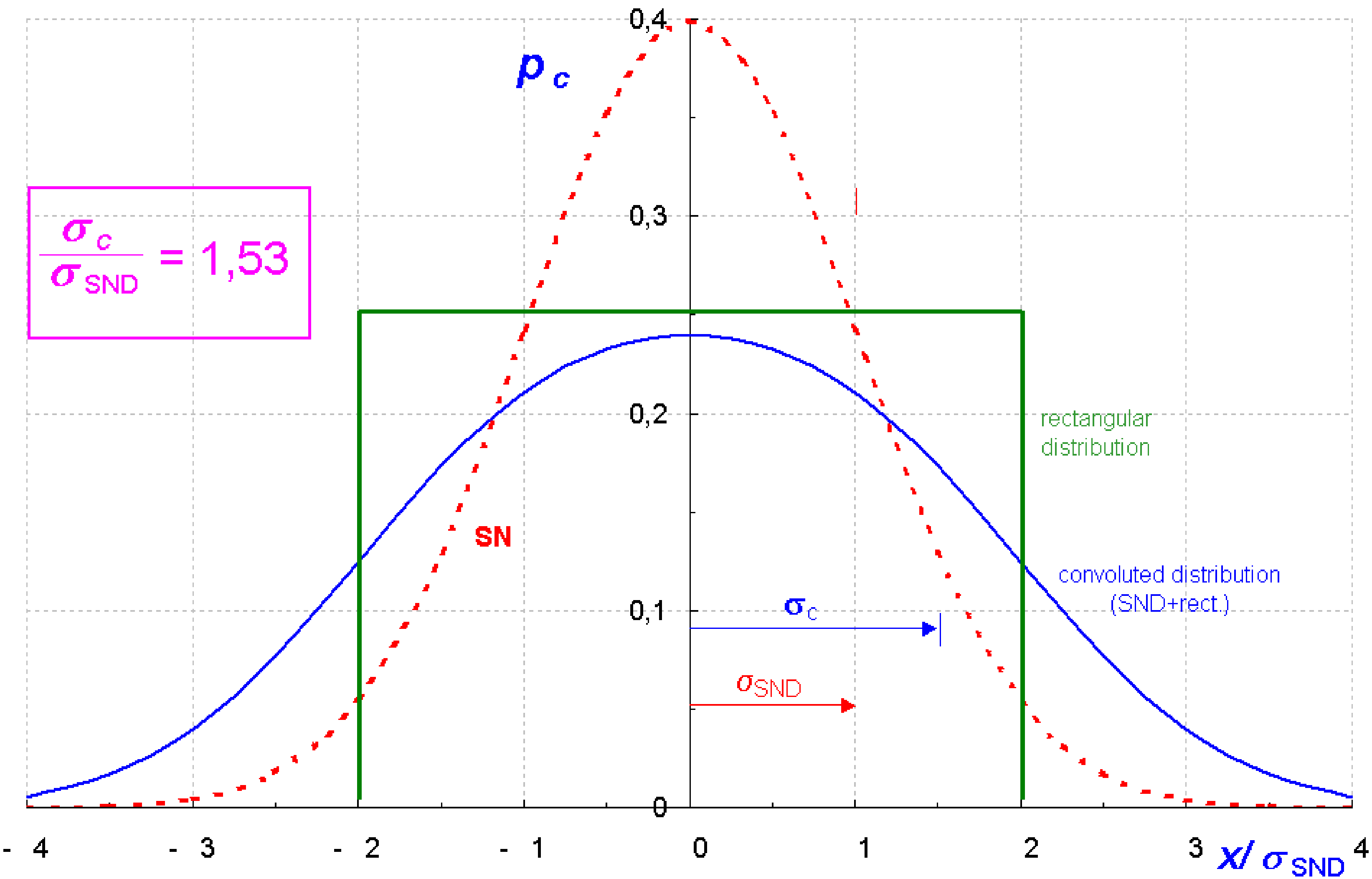


Convolution of a SND and a small rectangular distr.

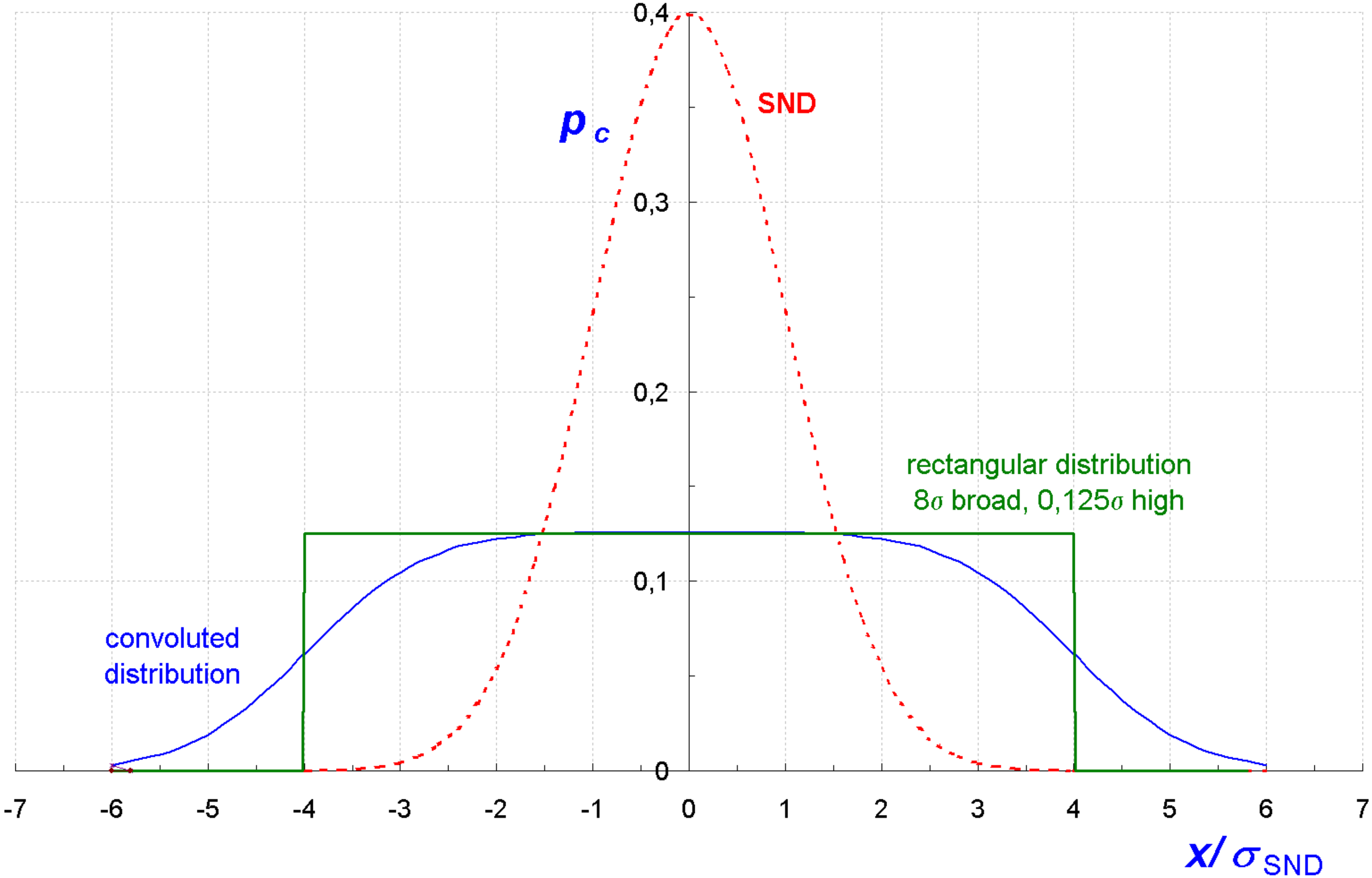


$$\frac{\sigma_c}{\sigma_{NV}} = 1{,}15$$

Convolution of a SND and a broad rectangular distr.



Convolution of a SND and a very broad rectangular distr.



What distribution of the convoluted function is the correct one?

GUM says:

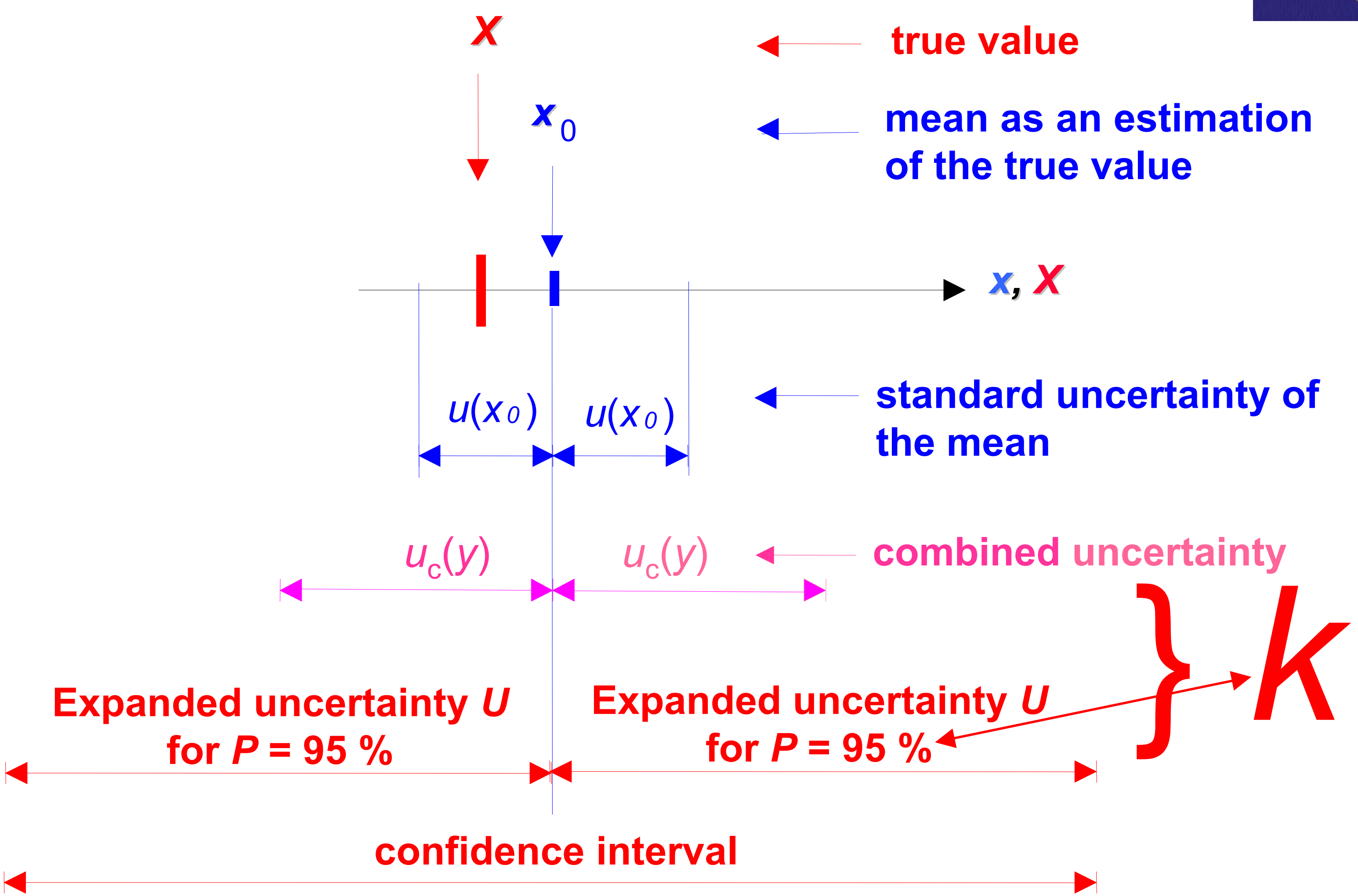
the distribution of the result is nearly a SND!

I think:

In practice this is a bad approximation, because we need an accurate extension factor *k*

- **The best way is to evaluate *k* by the method of convoluting**
- **the next way is to use the „Monte-Carlo-method"**
- **the „Method of effective freedoms" is not as good**
- **bad is $k = 2$**

Aim of evaluation: the expanded uncertainty



In the most cases the expanded uncertainty is too small, because

- we don´t know all contributions to uncertainty
- we don´t know all correlated effects

A great problem is the correct determination of the coverage faktor *k*:

- In the most uncertainty determinations $k = 2$ will be chosen. If you remember the pictures of the convoluted distributions it is very difficult to evaluate the correct expanded coverage factor, because the underlying distribution is only known in an approximative form.

- Example: The control of room temperatur in a range of $\pm \Delta t$ is associated with an **unknown distribution**. The easiest form is a triangle distribution. An other form is nearly the SND. But, what of these distributions are the correct one?

Summary

What is the uncertainty of uncertainty determination now?

<u>I guess: not less than ± 10 %</u>